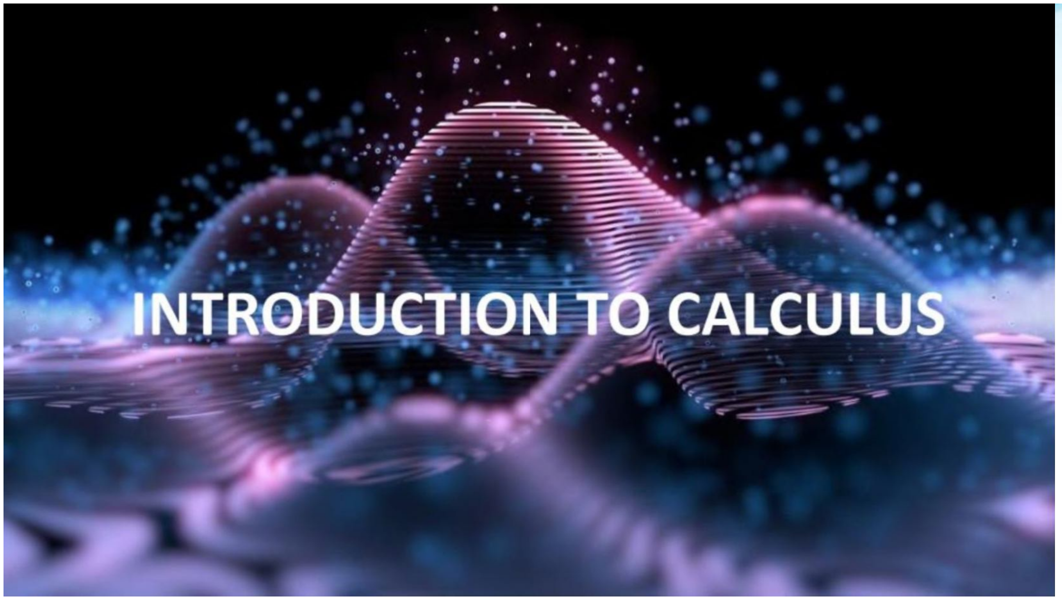


Calculus of real variables

Calculus of real variables

Differential Calculus:
- → Limits continuity, Differentiability
- → Mean value theorem
- → Taylor Series
- → Application of derivative
- → Partial Derivative

Integral Calculus:
- → Single Integral
- → Double Integral
- → Triple Integral

Variable

A variable is a mathematical symbol that represents a mathematical object. A variable may represent a number, a vector, a function a set or an element of a set. complex variable

Real variable	Vector Variable	Complex Variable
x is real variable	$\vec{r}: x\hat{i} + y\hat{j} + z\hat{k}$	$z = x + iy$
→ x can take all real values	$-\infty < x < \infty$	$-\infty < x < \infty$
$-\infty < x < \infty$	$-\infty < y < \infty$	$-\infty < y < \infty$
	$-\infty < z < \infty$	z = complex variable
	$\vec{r}$ is a 3D. -vector	

Definition of Set

A set is a well defined collection of distinct objects.
Sets are usually denoted by capital letters A, B, C, X, Y etc.

Example of Sets

(i) Set of all Natural Numbers.
(ii) Set of vowels in alphabet of English language.
(iii) Set of all states in India.

Elements of Set

belongs to

Consider a set $A = \{1, 2, 3, 4, 5\}$
Here, 3 is an element of set A and is represented mathematically as $3 \in A$.
If x is an element of set A then we write $x \in A$. If x is not an element of set A then we write as $x \notin A$.

Representation of Set

Set can be represented by two methods:-
(i) Roster Form:- In this elements are enclosed in curly brackets after separating them by commas.
1) Order of elements of a set is immaterial eg- {a, b, c}, {b, c, a} both denote same set.
2) An element is written only once in a set i.e. the set {1, 1, 2, 3} is same as {1, 2, 3}.

(ii) Set Builder method:- In this method we write staging properties which its elements satisfy
$A = \{x: P(x)\}$ or $A = \{x\ P(x)\}$
A is set of elements of x, such that x has property P.
Example:- $A = \{x \in N; x \leq 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

- **Null Set:-** A set having no element is called as a Null set. It is denoted by ϕ or { }. Null set is subset of every set.
  Ex:- $\{x: x \in N, 4<x<5\} = \phi$

- **Subset:-** If every element of set A is also an element of set B, then A is called subset of B and we write it as $A \subset B$.

$$A \subset B \Leftrightarrow [x \in A \Rightarrow x \in B]$$

Ex:- If $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$ then $A \subset B$.
- If a set A has n elements then total number of subsets of set A are 2^n.

a set is also a subset of itself.
null set is also a subset.

Ordered Pairs

If a set A and $a, b \in A$ then ordered pair of elements a and b in A is denoted by (a, b), where a is called first coordinate and b is second co-ordinate.
Note:- Ordered pairs (a, b) and (b, a) are different.

ordering is important.

Cartesian Product of Two sets

Cartesian Product of two sets A and B is set of all those ordered pairs whose first co-ordinate belongs to A and second co-ordinate belongs to B. This set is denoted by $A \times B$.

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$$

If A has p elements and B has q elements then $A \times B$ has pq elements.

Ex:- If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ then

$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

Note:- $A \times B \neq B \times A$.

Relations

A relation R from non-empty set A to another non-empty B is a subset of $A \times B$.

$R \subset A \times B$

$R \subset \{(a, b) : a \in A, b \in B\}$

- If (a, b) be an element of R then we write aRb i.e. $(a, b) \in R$.
- If (a, b) is not an element of R then we write $a \not R b$ i.e. $(a, b) \notin R$.

Example:- Let $A = \{1, 2\}$ and $B = \{3\}$ then,

$A \times B = \{(1, 3), (2, 3)\}$

$R = \{(1, 3)\}$

Important Points of an Relation

(1) If A has m elements and B has n elements then total number of different relations from A to B is 2^{mn}.

cartesian product has mn ordered pairs.

Subset = 2^{mn}

(2) If $R = A \times B$ then Dom R = A and Range R = B.

(3) Domain as well as range of empty set is ϕ.

(4) If A = domR and B = ranR then we write B = R[A]

$A = \{1, 2, 3\}$ $B = \{4, 5\}$

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$

$R_1 \subset A \times B = \{(1,4), (2,5), (3,4)\}$

domain = $\{1,2,3\}$ range $\{4,5\}$

$R_2 = \{(1,4), (2,4)\}$

domain = $\{1,2\}$

range = $\{4\}$

Function

Functions are a tool for describing real world in mathematical terms.

Eg- Temperature at which water boils depends on elevation above sea level, distance an object travels depends on elapsed time.

In each case value of one variable quantity say 'y' depends on value of another variable quantity which we often call 'x'.

We say 'y' is a function of ' x ' and write it symbolically as :

$$y = f(x)$$

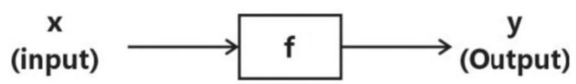
dependent variable

independent variable

Dependent and Independent Variable

The symbol f represents the function.

The letter ' x ' is independent variable representing input value of f and y is dependent variable or output value of f at x.

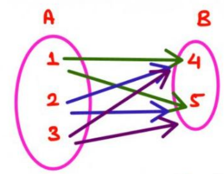


Functions

**
SN

Let D and Y be two non empty sets.

A function f from a set D to set Y is a rule that assigns a unique value f(x) in Y to each x in D.

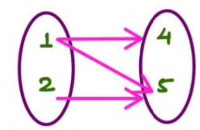


Cartesian product

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$

relation

$R = \{(1,4), (1,5), (2,5)\}$

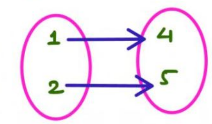


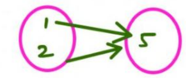
domain range

function

$f = \{(1,4), (2,5)\}$

$f = \{(1,5), (2,5)\}$



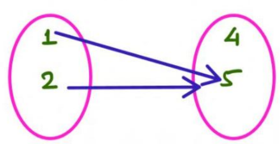


Domain, Co-domain and Range

**
SN

Domain:- The set of D is called domain of f.
Co-domain:- The set of Y is called co-domain of f.
Range:- The range of f is denoted by R_f and is set consisting of all images of elements of domain D.

- Range of f = {f(x): x∈D}
- Range of f is always subset of co-domain Y.



domain D= {1,2}

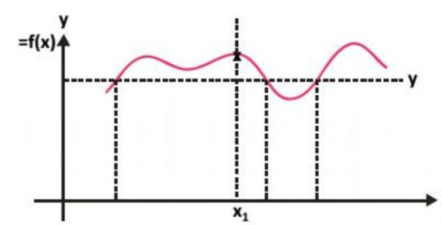
co-domain = {4,5}

range= {5}

Graphical Representation

The graph of y = f(x) where f is a function, consists of points whose coordinates (x, y) satisfy (x, y) satisfy y = f(x).

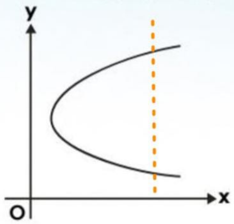
Also, it is such that lines drawn parallel to the y-axis do not intersect the curve at more than one point.



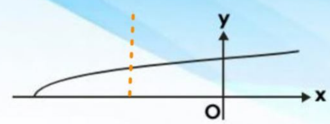
Examples

Which of following graphs represent a function

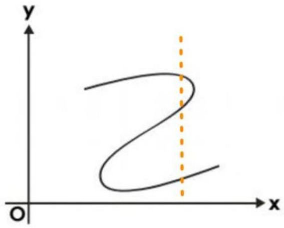
(a)



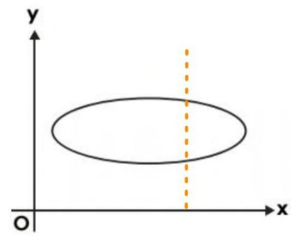
(b) ✓



(c)



(d)



Examples

→ value of x at which function value (y) is finite

Determine domain and range of given functions-

(A) $f(x)=\frac{1}{x+3}-5$ domain = $R-\{-3\}$ range = $(-\infty,\infty)$
↳ open interval $-\infty$ & $+\infty$ not included

(B) $F(x)=\sqrt{5x+10}$ domain : $5x+10 \geq 0$
$5x \geq -10$
$x \geq -2$ $[-2,\infty)$
range : $[0,\infty)$
↳ closed interval '0' is included

(C) $g(x)=\sqrt{x^2-3x}$

$x^2-3x \geq 0$
$x(x-3) \geq 0$
$x \geq 0$ and $x \geq 3$ i.e. $x \geq 3$
$x \leq 0$ and $x \leq 3$ i.e. $x \leq 0$
→ domain $(-\infty,0] \cup [3,\infty)$
$= R-(0,3)$
range = $[0,\infty)$

Open and Closed Intervals **SN

Open-interval:- Value of boundary of interval is not included inside interval. Represented by round bracket.

Closed interval:- Value at boundary of interval is included inside interval. Represented by square bracket.

$x \in (a, b) \rightarrow$ x lies between a and b but x cannot be equal to either a or b.
↑
belongs to

$x \in (a, b] \rightarrow$ x lies between a and b $x \neq a$ but x can be equal to b.

open closed

Functions of real variables

Standard Functions:

→ Algebraic
- → Polynomial
- → Rational
- → Irrational
- → Piecewise
  - → Modulus
  - → Signum
  - → Greatest Integer Function
  - → Fractional part function
  - → Least integer function

→ Transcendental
- → Trigonometric
- → Exponential
- → Logarithmic/Inverse of exponential
- → Inverse trigonometric curves

POLYNOMIAL FUNCTIONS

$y = f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ where $a_0, a_1, a_2, \ldots$ are real constant and n is a non-negative integer.

f(x) is called as polynomial function of degree = n. ← highest power of 'x'

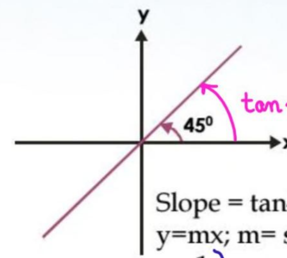
Eg: x^3+4x^2+3x+2

Polynomial of degree 3

- All polynomial functions are algebraic function but converse is not true.

Polynomial Functions

$y = x$: Identity Function

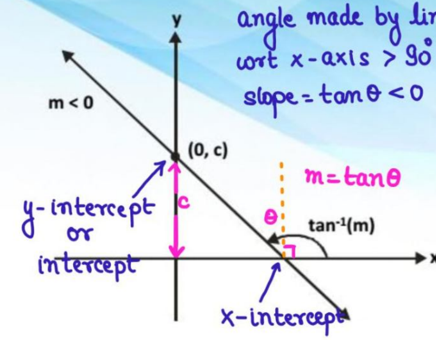


Slope = tan45 = 1

y=mx; m= slope

m=1 → passes through origin

tan of angle made by line wrt x-axis



angle made by line wrt x-axis > 90°

slope = tanθ < 0

Line having negative slope m and positive intercept c

Slope intercept form: $y = mx + c$

Equations of a Straight Line

① Slope-intercept form

$y = mx + c$

$m = \text{slope} = \tan\theta$

θ = angle wrt x-axis

c = y-intercept

② Slope-point form

slope = m

line passes through (x_0, y_0)

$(y - y_0) = m(x - x_0)$

③ 2 point form

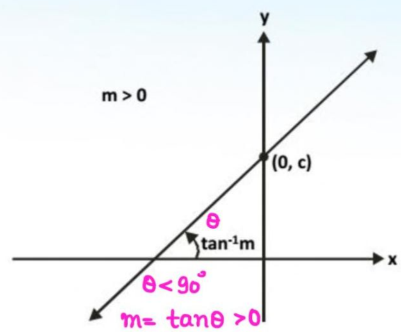
line passes through (x_1, y_1) & (x_2, y_2)

slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$(y - y_1) = m(x - x_1)$

$(y - y_1) = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

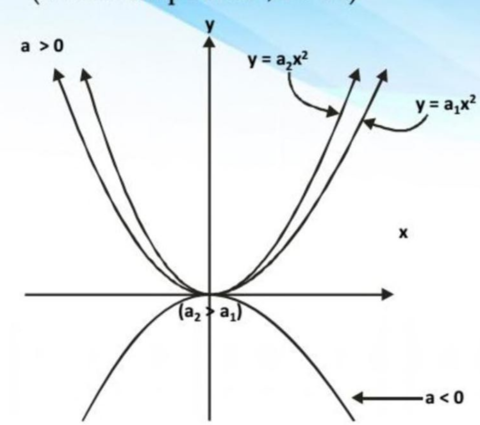
Polynomial Functions

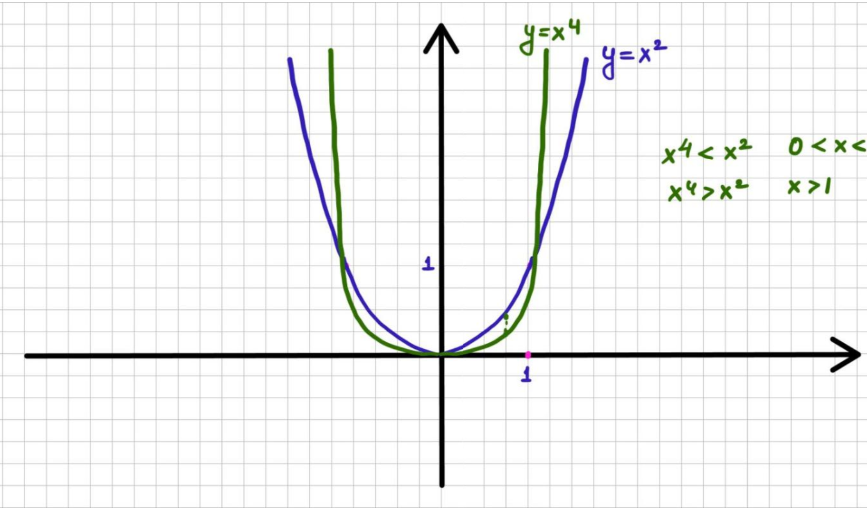


Line having positive slope m and positive intercept c

$y = ax^2$

(similar shape of ax^4, ax^6 …)





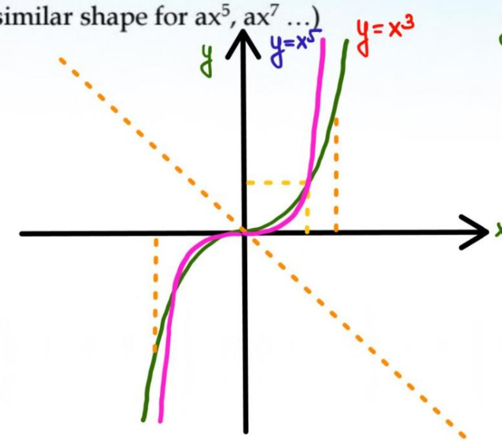
$x^4 < x^2 \quad 0 < x < 1$

$x^4 > x^2 \quad x > 1$

Polynomial Functions

$y = ax^3$

(similar shape for ax^5, ax^7 …)



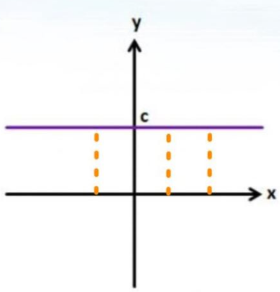
symmetrical in 1st & 3rd quad.

symmetric about line $y=x$ or $y=-x$

such functions are called odd function

$f(-x) = -f(x)$

CONSTANT FUNCTION



y = c = constant

Many-one function

→Multiple inputs give same output

- Polynomial of degree 0

functions allow many-one mapping but not one-many mapping.

TEST YOURSELF!

Draw the graphs of following functions

1. $y = x + 1$ $\qquad y = mx + c \quad m = 1 = \text{slope}$

$\theta = 45°$

$c = 1$: intercept

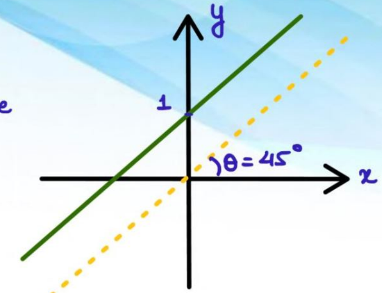
2. $y = x(x-1)(x-2)$

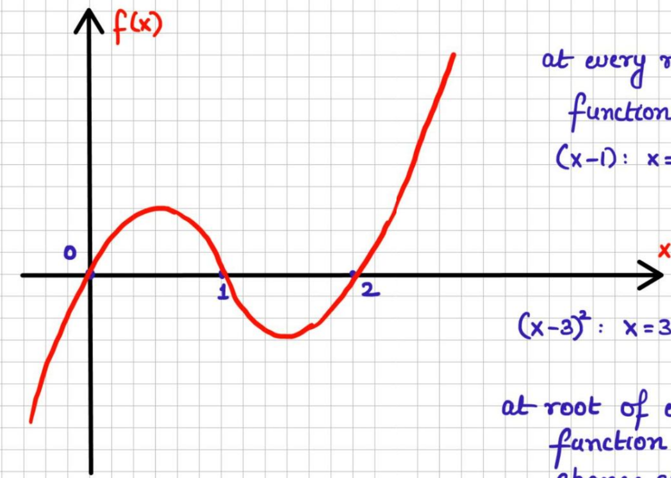
② roots : $y = 0$

$x = 0 \qquad x < 0 \rightarrow (x-1) < 0$

$x = 1 \qquad (x-2) < 0$

$x = 2 \qquad y < 0$





at every root of order = odd function changes sign

$(x-1)$: $x=1$ is a root of order = 1

$(x-3)^2$: $x=3$ is a root of order 2

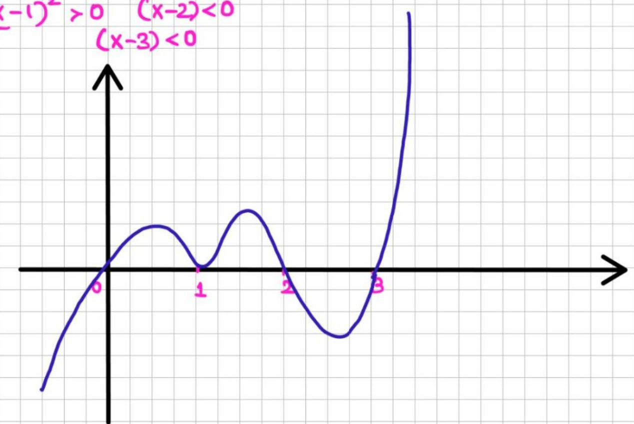
at root of even order function does not change sign

$y = x(x-1)^2(x-2)(x-3)$

roots : 0, 1, 2, 3

$x<0$ $\quad (x-1)^2>0 \quad (x-2)<0$
$(x-3)<0$

$y<0$



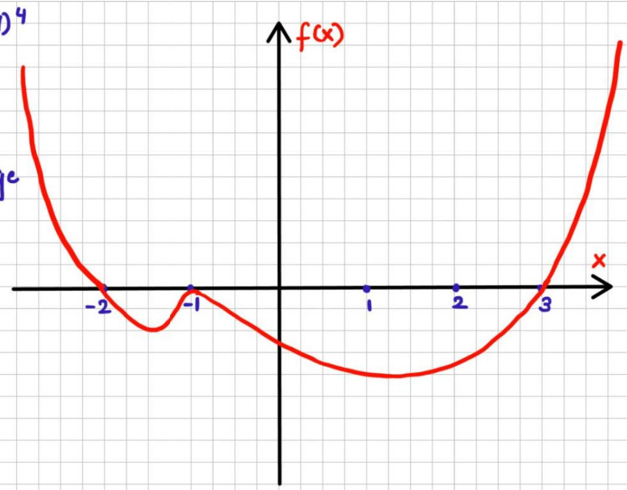
$y = (x-3)^3(x+2)^5(x+1)^4$

roots : $-1, -2, 3$

$\downarrow$ even, $\downarrow$ odd, $\downarrow$ odd — $\underbrace{\text{odd} \quad \text{odd}}_{\text{sign change}}$

$x<-2$ $\quad (x-3)^3<0$
$(x+2)^5<0$
$(x+1)^4>0$

$y>0$



RECTANGULAR HYPERBOLA

$y \propto \frac{1}{x}$

as x increases y reduces

$\because xy = a \quad (a>0)$

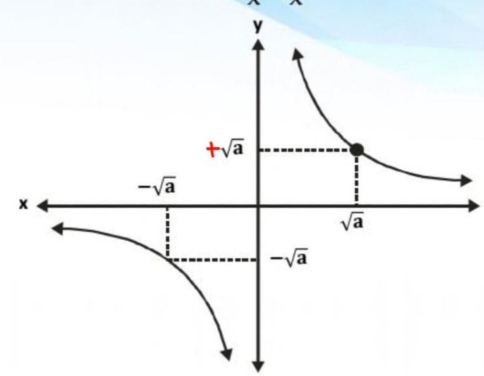
$x>0 \quad y>0$: 1st quad

$x<0 \quad y<0$: 3rd quad

domain = $R - \{0\}$

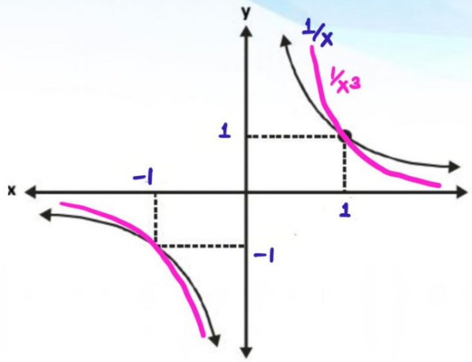
range : $(-\infty, \infty)$

$y = a/x$ or $xy = a$
(similar shape for $\frac{a}{x^3}, \frac{a}{x^5} \ldots$)



RECTANGULAR HYPERBOLA

$y = a/x$ or $xy = a$
(similar shape for $\frac{a}{x^3}, \frac{a}{x^5} \ldots$)



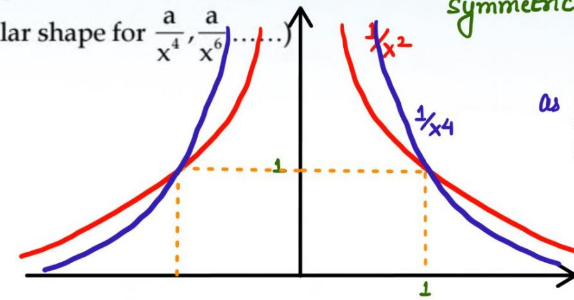
IRRATIONAL FUNCTIONS

$y = \frac{a}{x^2}$

(similar shape for $\frac{a}{x^4}, \frac{a}{x^6} \ldots\ldots$)

$f(-x) = \frac{a}{(-x)^2} = \frac{a}{x^2} = f(x)$: even function

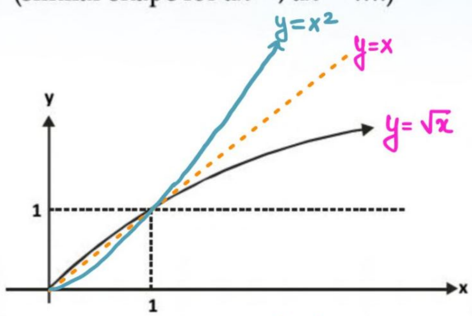
symmetric about y-axis



as power ↑
value of function for $x<1$ ↑
value for $x>1$ ↓

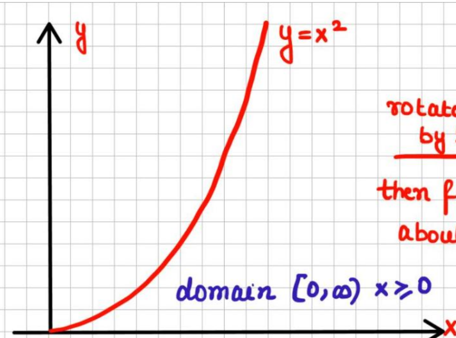
Square Root Function

$y = ax^{1/2}$
(similar shape for $ax^{1/4}$, $ax^{1/6}$ ….)

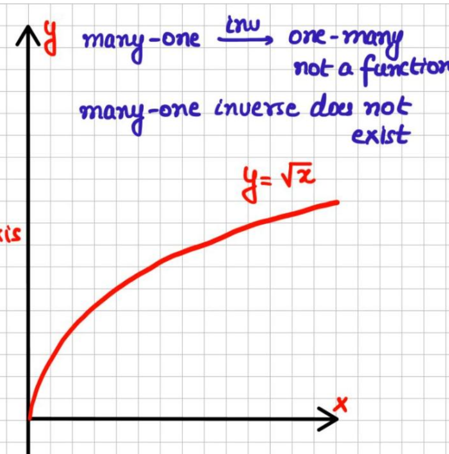


$\sqrt{x}$ is reflection of x^2 about $y=x$ line

- square root or reciprocal of even power $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ are always defined for $x>0$
  domain: $x>0$
- square root is an inverse of square function.
  $2^2 = 4$ $\sqrt{4} = 2$
  $5^2 = 25$ $\sqrt{25} = 5$



rotate ACW by 90° → then flip about y-axis



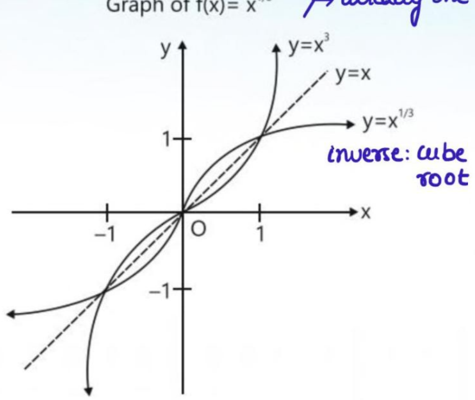
many-one $\xrightarrow{\text{inv}}$ one-many not a function

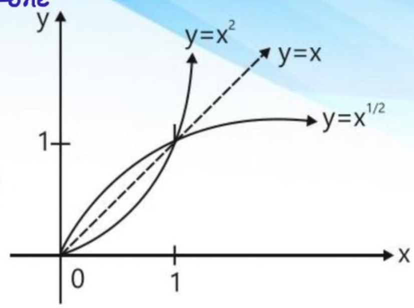
many-one inverse does not exist

to derive inverse of a function we must restrict domain of original function s.t. it is one to one

IRRATIONAL FUNCTIONS

Graph of $f(x)= x^{1/3}$ → already one-one





Modulus Function

$y=|x| \rightarrow$ Graph of $f(x)=|x|=\begin{cases} -x : x<0 \\ 0 : x=0 \\ x : x>0 \end{cases}$

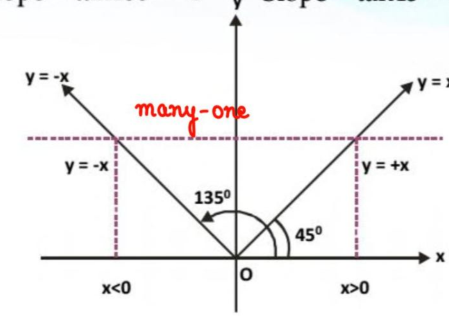
$|-3| = 3 = -(-3)$ $|3| = 3$

$|-6| = 6 = -(-6)$ $|6| = 6$

$x<0$ $-x$ $x>0$ x

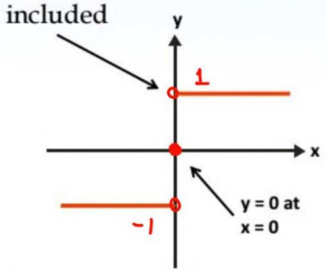
Whenever we take modulus we get positive value or magnitude of a value.

Slope = tan135 = -1 Slope = tan45 = 1



SIGNUM FUNCTION

$x = 0$ is not included



$$\operatorname{sgn}(x)=\frac{|x|}{x}, x \neq 0$$
$$=0, \ x=0$$

$$|x|=\begin{cases} x, & x>0 \\ -x, & x<0 \\ 0, & x=0 \end{cases}$$

$$\operatorname{sgn}(x)=\begin{cases} 1, & x>0 \\ -1, & x<0 \\ 0, & x=0 \end{cases}$$

EXAMPLES

Solve the following equations:

(1) $|x| = 5$ $x=5$ & $x=-5$

(2) $|x| = -5$ no solution

(3) $|x| < 5$ $-5 < x < 5$

↳ mathematical $x<5$

$-x < 5$ multiply - on both sides

$x > -5$

solution $-5 < x < 5$

$x \in (-5, 5)$